Analysis for robustness to features of network structures

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Abstract

The researches in scale free (SF) network have shown that degree assortativity is related to robustness. However, little attention has been given to the relation between assortativity and degree distribution. Therefore, this paper proposes the model which can regulate network assortativity and degree distribution. Based on this model, we analyze the feature about robustness by a computer simulation. The result shows some cases that robustness decreases when assortativity increases. In addition, the result demonstrates that the maximum degree has influenced in robustness strongly.

Keywords: complex network, assortativity, robustness.

1. Introduction

Over the past decade, there has been a lot of research relating the structure and function of networks, such as acquaintance relations, the Internet, the WWW, and interaction of protein in the living body and so on. The almost of these networks are social infrastructures, and there are common features in the above networks [1].

The features are that the frequency distribution for the degree (the number of links linked to a node) k of a node follows power law $p(k) \sim k^{-\gamma}$. The power law of degree distribution expresses the feature in which a few high-degree nodes and many nodes of low-degrees are intermixed. For the network with degree distribution of power law, the average degree of node isn't a representative parameter of network structure.

Therefore, such networks are called scale free (SF) network. Moreover, since SF networks include hubs (nodes with high-degree), the network includes the following features:

1. robust against random failure (removing nodes selected randomly).

2. vulnerable against hub attack (removing hubs intendedly).

To random failure, connectivity of SF network can be maintained, because a few hubs connect to other low-degree nodes and are less likely to be broken, while many links will be disconnected by removing a few hubs out of a network. Therefore, SF network has the above-mentioned features of 1. and 2.

It was shown clearly that the robustness to the hub attack on SF network is influenced not only degree distribution but in the relation of the degree of an adjacent node in research of SF network [1, 2].

Moreover, an actual network has the feature that the relation of the degree of adjacencies nodes changes with network structures. One of the measurements to show the relation of degrees between adjacent nodes has Assortativity. In actual social networks including networks, such as scientists, film actors, and business administrators, there is tendency which nodes with the same degree connect, and such networks have positive assortativity (Assortative). Meanwhile, in the Internet and the WWW, there is tendency which nodes with different degree connect, and such networks have negative assortativity (Disassortative).

It has been proved by both simulations and theoretical analysis that Assortative networks have high robustness against hub attack [2, 3]. However it has been also proved that some of Assortative networks are not robustness [5, 7].

While research of the network feature is done briskly, there are features which are not analyzed about assortativity to the size (the number of nodes) of networks and the degree distribution.

So, in this paper, to construct robust networks, we analyze robustness to the above parameters by computer simulation using a model to generate networks, which can regulate assortativity.

2. Measurement to relation of degrees between adjacent nodes

The relation of the degree of adjacencies nodes is different by network structures. Then, degree assortativity (assortativity) is proposed as the measurement which shows the relation of the degree between adjacent nodes in a network.

assortativity is written as r and defined by the following formula.

$$r = \frac{m^{-1} \sum_e k_i(e)k_j(e) - \{m^{-1} \sum_e \frac{1}{2} [k_i(e) + k_j(e)]\}^2}{m^{-1} \sum_e \frac{1}{2} [k_i(e)k_j(e)] - \{m^{-1} \sum_e \frac{1}{2} [k_i(e) + k_j(e)]\}^2} \quad (1)$$

Where the number of links of the whole network is m , the degree of the adjacent node is $k_i(e)$ and $k_j(e)$. r takes $[-1, 1]$.

In the case of $r > 0$, networks are assortative. In the case of $r \rightarrow 1$ networks tend to connect nodes with the same degree.

In the case of $r = 0$, networks are uncorrelated. That is, the connection probability between the nodes i and j becomes unrelated to the degree difference of i and j .

In the case of $r < 0$, a network has negative assortativity. In the case of $r \rightarrow -1$ networks tend not to connect nodes with the same degree.

Assortative SF network is robustness. In other words, it is shown that robustness against hub attack increases.

The following chapter explains the measurement of network robustness.

3. Robustness in SF network

SF network, where a few very high-degree nodes and many low-degree nodes are intermingled, can keep connection to random failure in disaster (Robustness is high).

The reason is that many nodes keep connection while hubs are not destroyed. This is because probability of hubs destroyed by random failure is very low, since rate of hubs to the whole network is low.

Although they have influence to the adjacent nodes, when low-degree nodes are destroyed, the influence to the whole network is very small. In other words, the network cannot keep connection by a few hub attacks, and it becomes dysfunction.

One of the measurement of robustness is $s(q)$ that the ratio of largest connected component (LCC) $S(q)$ (size of LCC) to N in the size of whole network: $s(q) = S(q)/N$.

Moreover, the average of the connected component of

nodes except LCC is the measurement, too (Average size of islands : $\langle s \rangle$). Where q is fraction of hub attack .

The above-mentioned measurements express the connectivity to q , however they cannot express the robustness of the whole network. Then, R value which is measurement of the robustness in the whole network has been proposed. The formula is shown in the following [2]:

$$R = \frac{1}{N} \sum_{q=1/N}^1 s(q). \quad (2)$$

R value is total of $s(q)$ divided by N of size of the whole network. The range of R value is $[1/N, 0.5]$.

The values of each measurement to relation of robustness are as follows.

First, if $s(q)$ is the larger, network is the more robust. Moreover, since it is thought that the LCC was divided when $\langle s \rangle$ takes peak values, if, when $\langle s \rangle$ takes peak values, q is large, the network is robust. In addition, since R value is provided by integration of $s(q)$ to q , a network which has high R value is robust.

The following chapter explains the network model used for our simulations.

4. Generation model of networks, which can regulate assortativity

The Degree distribution is selected and generates the regular network of each degree. Then, undirected-links between each network are rewired. The method is shown in following.

(Initialization)

1. N is selected.
2. The degree distribution $P(k)$ is selected based on exponent γ , minimum degree k_{min} , and maximum degree k_{max} .
3. Based on N and $P(k)$, the number $N(k)$ of nodes of the degree k ($k=k_{min}, k_{min+1}, \dots, k_{max}$) is selected.
4. The regular graph is generated by the nodes of the same degree k .

(Rewiring of links)

1. Based on the rewiring probability p of link, each of the degree k and $k+1$ node is selected single node, and it is considered as n_k and n_{k+1} , respectively.
2. However, if n_k has already connected with n_{k+1} , select n_k and n_{k+1} again.
3. Randomly, one of each link of n_k and n_{k+1} is selected, and it is removed.

Table.1 : the parameter of the network model

Parameter	valuable	value
Number of nodes	N	1000,2000,3000, 4000, 5000
Power	γ	2.0, 2.5, 2.75, 3.0
Minimum degree	k_{min}	3
Maximum degree	k_{max}	5, 6, 10
Rewiring probability of links	P	0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.9
Number of samples	M	20
Failure ratio	q	0 ~ 1
step size of a failure ratio	Δq	0.01

4. n_k and n_{k+1} are connected.

In this model, if the rewiring probability of link is high, assortativity will decrease.

5. Simulation analysis of robustness to the parameter of network models

First, parameters of the network model used for following simulations are shown in Tab.1.

To each network model, failure rate q was changed between 0~1 by the step width $\Delta q=0.01$. The method of node removal is as follows:

1. Select a single node with degree k_{max} .
2. When there are two or more nodes with k_{max} degree, select randomly a single one from them.
3. The selected node is removed from the network (the degree of each node adjacent to the removed one decrease 1).

Because the above process 3, the order of nodes in degree may change. Since, whenever it removes a single node, nodes with k_{max} degree are checked.

Moreover, there may be two or more nodes with k_{max} degree. Therefore, a single node removed from nodes with k_{max} degree is selected randomly. The removed nodes may change in each simulation for this operation. Therefore, we show results averaged in 10 experiments for each combination of parameters.

Figures from 1 to 5 show experimental results for robustness.

Fig.1 shows following result about $S(q)/N$ to each parameter. N of network size seldom affects $S(q)/N$. It is likely that the reason is because degree distributions follow power law and have scale free property. In other words, it is because the network features is invariant regardless of the number of nodes.

Moreover, when p , probability of rewiring links, is high, failure ratio when networks are divided is low. This shows influence of hub attack increases, when hubs connect to low-degree nodes with high frequency.

Next, when k_{max} increases, robustness increases. The reason is that due to degree distribution in power law, when k_{max} increases, ratio of nodes with k_{min} degree decreases, which is the great portion of S .

Furthermore, when γ increases, robustness decreases because ratio of nodes with k_{min} degree increases.

Fig.2 shows following results about $\langle s \rangle$, average size of islands to each parameter. When p increases, the peak value of $\langle s \rangle$ decreases, since it is necessary to remove many nodes in order to divide networks, when p is high. Thereby, $\langle s \rangle$ when networks are divided is small.

Fig.3 shows results about R value to each parameter. Originally, R value is sum of $S(q)/N$ every removing a single node. However, the results in Fig.3 are approximation by numerical integration $s(q)$ for q with step width $\Delta q=0.01N$.

First, when γ increases, R value decreases. Second, when k_{max} increases, R value increases regardless of the number of nodes. Third, R value increases, when p is small.

It is thought that these reasons are same for reasons with S/N since R value corresponds to S/N .

Fig.4 shows following results about assortativity to each parameter. Assortativity is high when k_{max} is high, and the higher p is, the higher the increasing ratio of assortativity is, because probability that high-degree nodes can connected increases, and it is obvious by the definition. With $N = 5000$, toward $k_{max} = 5$, and 6, assortativity with $\gamma = 2$ is lower than with $\gamma = 3$. However, assortativity increases toward $k_{max} = 10$, because assortativity is affected by degree of node, and the number of nodes with arbitrary degree.

(a) $N=5000$, $\gamma=2$, $k_{\max}=5$

(b) $N=5000$, $\gamma=2$, $k_{\max}=10$

(c) $N=5000$, $\gamma=3$, $k_{\max}=5$

(d) $N=5000$, $\gamma=3$, $k_{\max}=10$

Fig.1 q toward S/N

Fig.2 q toward $\langle S \rangle$
 $N=5000$, $\gamma=2$, $k_{\max}=5$

Fig.3 $k_{\max}$ toward R
 $N=5000$, $\gamma=2$

Fig.4 $k_{\max}$ toward r
 $N=5000$, $\gamma=2$, $k_{\min}=3$

$N=1000$, $\gamma=3$

$N=5000$, $\gamma=3$

Fig.5 r toward R

Fig.5 shows a result about R value to assortativity. Mostly, when assortativity increases, R value tends to increase. However, when $k_{\max}$ is low and N is high, it shows cases that R value decreases. This tendency comes out more

notably when γ is high.

6. Conclusion

We proposed the network generation model which can regulate assortativity. Then, the hub attack experiment was conducted to the artificial network. The result is shown in following.

As main results, it turns out that robustness is affected in degree distribution strongly. Specifically, when the maximum degree is high, robustness and assortativity are high.

Moreover, when the exponent of degree distribution is high, robustness is low, while assortativity is hardly affected.

In addition, when rewiring probability of links increases and assortativity is high, failure ratio to divide networks is low.

From these results, it turns out that the maximum degree affects robustness and assortativity in SF networks.

Furthermore, we find out that some Assortative networks are not robustness. This result is clarified by the previous study, too [6].

As future subject, we think a case that assortativity is less than simulations of this report, and extend our framework to directed-graphs.

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