Adaptive Waveform for Cognitive Radar to Minimize Cramér-Rao Bound

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Abstract

This study concern with a Cognitive radar operating in an environment with very low signal to noise ratio and multiple interested targets. Cognitive radar is an intelligent parameter estimation system that the transmitted signal depends on the input signal. The target parameters are difficult to be estimated when signal-to-noise-ratio is very low. Therefore, retransmitting a signal that gives better parameter estimation is required. In this study, adaptive waveform design, a technique to design the transmitted waveform from the prior information, is proposed. The waveform is designed to minimize the mean-square error of the estimated parameters. The mean-square-error is minimized by minimizing the Cramér-Rao Lower Bound. Since the objective function is non-linear, a stochastic global optimization technique (particle swarm optimization) is introduced. To improve the estimation accuracy, an algorithm to partition posterior probability density function (pdf) of each target is added. The algorithm reduces the redundancy information by reducing the side-lobe of the posterior pdf of each target. We can prove that the performance (accuracy) of the proposed method is better than the existing method.

Keywords: Adaptive waveform, Cognitive, Cramér-Rao bound, Bayes' theorem, Particle swarm optimization.

1. Introduction

Cognitive radar (shown in Figure 1) is a Multi-input multi-output (MIMO) system, which the collected data is passed through a feedback loop [1]. MIMO radar is the system that has a multiple receivers and multiple transmitters which emit independent waveform. MIMO system can be categorized into two architectures.

The first architecture is a system with widely spread antennas. The second architecture is a co-located antennas system. The advantage of co-located MIMO to single-input multi-output (SIMO) system is better parameter identifiability [2]. Higher parameter identifiability offers more degree of freedom, lower power side-lobe at the antennas, and more variety of the transmitted waveform.

The waveform is one of the most important features of the radar. It determines resolution and accuracy of the estimated parameter. As a result of weak signal-to-noise-ratio, transmitting an orthogonal waveform (uniform beam pattern) fails to estimate the parameters, a new signal has to be sent to improve the estimation. It is designed for giving a better parameter estimation. The traditional adaptive method [3] proposed the technique to determine the transmitted waveform by minimize the Cramér-Rao bound (CRB) or Reuven-Messer bound (RMB) from prior information. The technique achieves higher estimation accuracy than the other existing signal transmission methods. However, when multiple targets are interested, the side-lobe of targets posterior probability density function (pdf) overlap each other. The performance is dropped, due to the expectation calculation in Fisher information matrix.

In this study, the algorithm to design waveform from prior information to minimize CRB is proposed. To maximize the estimation accuracy is to minimize the variance of an estimator expressed by Cramér-Rao bound. To achieve the higher estimation accuracy rate than the traditional method, the posterior pdf side-lobe is minimize by partitioning the pdf into each target. Between the transmitting step, a pdf of each parameter is re-estimated by using Least-square method [4]. Finally, the transmitted signal covariance matrix is the optimum solution of minimizing CRB.

Because the CRB objective function is nonlinear, the nonlinear optimization algorithm is applied. Particle swarm optimization (PSO) [5] is a heuristic search based on the collaborative behavior and swarming habits by some animals (such as birds). Particle swarm optimization is used because it has a few parameters to adjust and more computationally efficiency compare to other evolution search algorithm [6].

The paper is organized as follows. In Section II, the problem formulation is discussed. The proposed algorithm is described in Section III. Section IV, the algorithm is tested and compared to [3]. Finally, Section V is conclusion.

Fig. 1. Cognitive Scheme [3].

2. Problem Formulation

Consider a MIMO system with N_T transmitter and N_R receiver, at pulse step k , the system model is described by the Equation (1). The model is simplified in a Matrix form, Equation (2). The estimated parameters (θ) is the properties of the targets such as complex attenuation, angle of arrive, Doppler shift or propagation delay. The covariance matrix of the transmitted signal is shown in Equation (3). The noise of the system n is assumed to be multivariate normal distribution with mean equal to zero and covariance R_N .

$$x_{k,l} = H_k(\theta)s_{k,l} + n_{k,l} \quad (1)$$

where $x_{k,l}$: is the received signal at time index l ($x_l \in C^{N_R}$).

$H_k(\theta)$: is the system transfer function ($H(\theta) \in C^{N_R \times N_T}$).

θ : is the targets parameter vector ($\theta \in C^Q$).

Q : is the number of parameters.

$s_{k,l}$: is the transmitted signal at time index l ($s_l \in C^{N_T}$).

$n_{k,l}$: is the interference plus noise ($n_l \in C^{N_R}$).

$$X_k = H_k(\theta)S_k + N_k \quad (2)$$

where X : is $[x_1, \dots, x_l]$
 S : is $[s_1, \dots, s_l]$
 N : is $[n_1, \dots, n_l]$

$$R_S = SS^H \quad (3)$$

The goal of this study is to design S for achieving the highest accuracy of θ measurement.

3. Proposed Algorithm

3.1 System Mean-Square-Error

The measurement to determine the accuracy of an estimator is the mean-square-error (MSE). The system with lower MSE has more accuracy to detect a target. This subsection shows the derivation of MSE of the estimator.

The MSE of θ is shown in Equation (4). In Equation (5), the conditional MSE given the history input X_{k-1} is the mean square of the difference of true and estimated parameters when X_{k-1} is known. $\hat{\theta}$ denotes the measurement of θ .

$$\text{MSE} = E[(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T] \quad (4)$$

$$\text{MSE}_k|X_{k-1} = E_{X_k, \theta|X_{k-1}}[(\hat{\theta}_k - \theta_k)(\hat{\theta}_k - \theta_k)^T|X_{k-1}] \quad (5)$$

The MSE of the estimator can not go below the inverse of the Fisher information (J), this lower bound is called Cramér-Rao Bound [7]. In Equation (6), the conditional MSE is also greater than the inverse of conditional Fisher information. By using Bayes theorem [3], $p(X_k, \theta|X_{k-1})$ is equal to $p(X_k|X_{k-1}, \theta)p(\theta|X_{k-1})$. Therefore the conditional Fisher information matrix ($Q \times Q$) can be rewrite in Equation (7), where i, j are row and column of the matrix.

$$\begin{aligned} \text{MSE}_k &\geq J_k^{-1} \\ \text{MSE}_k|X_{k-1} &\geq J_k^{-1}|X_{k-1} \end{aligned} \quad (6)$$

$$\begin{aligned} J_{k,i,j}|X_{k-1} &= -E_{X_k, \theta|X_{k-1}}\left[\frac{\partial^2 \ln p(X_k, \theta|X_{k-1})}{\partial \theta_i \partial \theta_j} | X_{k-1}\right] \\ &= -E_{X_k, \theta|X_{k-1}}\left[\frac{\partial^2 \ln p(X_k|X_{k-1}, \theta)}{\partial \theta_i \partial \theta_j} | X_{k-1}\right] \\ &\quad - E_{\theta|X_{k-1}}\left[\frac{\partial^2 \ln p(\theta|X_{k-1})}{\partial \theta_i \partial \theta_j} | X_{k-1}\right] \end{aligned} \quad (7)$$

The CRB for general distributions has been derived in the Slepian Bangs formula [8]. Slepian Bangs formula is the method to compute the Fisher information

matrix for Gaussian distributed noise. Therefore, the Fisher information matrix (7) is derived into Equation (8). In Equation (8), only the terms that depend on the transmitted signal (X_k) or transmitted covariance matrix (R_{S_k}) is considered. ($\dot{H}_i = \frac{\partial H}{\partial \theta_i}$)

$$\begin{aligned} J_{k,i,j}|X_{k-1} &= -E_{X_k, \theta|X_{k-1}} \left[\frac{\partial^2 \ln p(X_k|X_{k-1}, \theta)}{\partial \theta_i \partial \theta_j} |X_{k-1} \right] \\ &= \\ -E_{\theta|X_{k-1}} [E_{X_k|\theta, X_{k-1}} &\left[\frac{\partial^2 \ln p(X_k|X_{k-1}, \theta)}{\partial \theta_i \partial \theta_j} | \theta, X_{k-1} \right] |X_{k-1}] \\ &= 2LE_{\theta|X_{k-1}} [S_k^H \dot{H}_{i,k}^H R_N \dot{H}_{j,k} S_k |X_{k-1}] \\ &= 2LE_{\theta|X_{k-1}} [tr(\dot{H}_{i,k}^H R_N \dot{H}_{j,k} R_{S_k}) |X_{k-1}] \end{aligned} \quad (8)$$

3.2 Minimizing Cramér-Rao Bound by Particle Swarm

The Cramér-Rao Bound equals the inverse of information matrix (Equation (8)). As the CRB is minimized, the MSE is minimized too. The transmitted signal (X_k) or transmitted covariance matrix (R_{S_k}) is optimally searched to minimize the CRB, as shown in Equation (9). To simplify vector to scalar cost function, the trace operation is used to the fisher information matrix [10]. The constraints in Equation (9) are referred to the power limitation and the properties of the covariance matrix, correspondingly.

$$\begin{aligned} R_S &= \arg \min_{R_S} \{tr(J_k^{-1})\} \\ \text{s.t. } R_{S,i,i} &= \frac{P_S}{N_T} \text{ for } i = 1, 2 \dots N_T \\ R_S &\succeq \mathbf{0} \\ R_S &= R_S^H \end{aligned} \quad (9)$$

The minimizing problem is taken by the particle swarm optimization [5]. The position of the particle (ϕ) is the value of the transmitted signal (X_k), where the magnitude of ϕ is fixed to $\sqrt{\frac{P_S}{N_T}}$ with any phase. The particles is moving in the N_T dimension space by the Equation (10). They are moving until a termination criterion is met (the number of iteration).

$$v_m(t+1) = v_m(t) + r_1[\widehat{\phi}(t) - \phi_m(t)] + r_2[\widehat{\phi}_m(t) - \phi_m(t)] \quad (10)$$

where t : is time step.

v_m : is the speed of the particle m .

r : is an uniform random variable (0 to 1).

ϕ_m : is the position of the particle m .

$\widehat{\phi}$: is historical position of the particle m that gives the minimum of Equation (8) (local).

$\widehat{\phi}_m$: is historical position of all particles that gives the minimum of Equation (8) (global).

3.3 Iterative PDF Calculation

If the position of transmitter and receiver are well designed, the maximum number of detectable targets (Q) can reach $\lceil \frac{N_T N_R - 1}{2} \rceil$ [2], $\lceil \cdot \rceil$ denotes the smallest following integer. By using this property, the ability to detect an object is maximized.

$$p(\theta|X_{k-1}) = \Gamma(p(\theta|X_{k-2})p(X_{k-1}|X_{k-2}, \theta)) \quad (11)$$

The posterior probability density function ($p(\theta|X_{k-1})$) is written by Bayes' theorem in Equation (11). To achieve CRB maximum-likelihood is used to compute $p(X_{k-1}|X_{k-2})$. $p(X_{k-1}|X_{k-2}, \theta)$ is based on joint multivariate normal distribution, shown in Equation (12). From joint multivariate normal distribution, $p(X_{k-1}|X_{k-2}, \theta)$ is simplified into one dimensional data by summing up all distribution of every target.

$$\begin{aligned} p(X_{k-1}|X_{k-2}, \theta) &= \\ \frac{1}{\sqrt{(2\pi)^{N_R} |R_N|}} &e^{-\frac{1}{2}(X_{k-1} - H_{k-1}S_{k-1})^H R_N^{-1}(X_{k-1} - H_{k-1}S_{k-1})} \end{aligned} \quad (12)$$

$$f(\theta) = \sum_{i=1}^{\lceil \frac{N_T N_R - 1}{2} \rceil} \frac{a_i}{\sigma_i \sqrt{2\pi}} e^{-\frac{(\theta - \mu_i)^2}{2\sigma_i^2}} \quad (13)$$

To minimize the pdf's side-lobe, the $\Gamma()$ is a function to partition the posterior pdf of each target. First, the pdf of each target is added together. Second, a Gaussian mathematical model is constructed as Equation (13). The number of term in the model is the maximum number of detectable targets ($\lceil \frac{N_T N_R - 1}{2} \rceil$), where a, μ and σ are constant, mean and standard deviation, correspondingly. Third, Least-square with multi-initial points is used to fit the added posterior pdf to the model. Finally, the pdf is spiting into each interested target by each term in the model. The example of partitioning is shown in Figure 2.

The overall system is described in Algorithm 1. All of the expectation calculations are taken by Metropolis-Hastings Monte-Carlo integration [9]. The proposal distribution of Metropolis-Hastings algorithm is equal to Gaussian distribution with mean and variance equal to mean and variance of $p(\theta|X_{k-1})$.

Fig. 2. The example of pdf partitioning. Top is the individual target's pdf. Middle is the summation target's pdf. Bottom is the rearranged pdf.

Algorithm 1 The adaptive waveform design method.

Initialization:

1. Set $k = 0$.
 2. Set all unknown pdf to uniform distribution.
 3. Calculate $p(\theta|X_k)$ from Equation (11).
- Loop: While user constraints are not satisfy.
1. $k = k + 1$.
 2. Calculate J from Equation (8).
 3. Find R_S from the optimization problem in (9).
 4. Construct S from R_S .
 5. Collect X_k from Model (2).
 6. Calculate $p(\theta|X_k)$ from Equation (11).
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4. Numerical Result

The performance of the proposed is simulated compare to the traditional method [3] via MATLAB. The simulation is running on a 3.6 GHz Intel i7 2700 with 8 GB of memory. There are 4 transmitters and 4 receivers. We assume that all parameters except the location of the target are known. Thus, the variable (θ) is a vector of target's location. The targets are assumed to locate at -20 and 45 degree. Signal to noise ratio is set to -6dB. The noise covariance matrix (R_N) is unknown, therefore it is assumed to be an identity matrix.

For the traditional CRB method, the initial value of prior pdf of the first target is set to uniform distribution

when $\theta < 0$ and zero when $\theta > 0$, and vice versa for the second target.

The resolution of maximum-likelihood estimator is 0.1 degree with 4 variables are allowed. The particle swarm optimization uses 100 particles to search Equation (9). The number of iteration is limited to 300.

Figure 3 is the posterior pdf of each transmitting step. At the 6th pulse step, the mean-square-errors of the proposed and traditional method are 0.4 and 3 degrees, correspondingly. Figure 4 shows the average MSE at each pulse step after 200 Monte-Carlo simulations. This algorithm takes about 5 seconds to compute one pulse step. Compare to the traditional CRB method and orthogonal waveform, the proposed method achieves significantly higher accuracy rate with about half a second more for Least-square fitting.

5. Conclusions

In the paper, a new adaptive method to design transmitted beam pattern is proposed in order to minimize CRB. With the partitioning posterior pdf step, the CRB can be accurately estimated. The accurate CRB results in a better waveform that matches to the targets. The numerical result illustrates that the proposed method has higher estimation accuracy at every pulse step.

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Fig. 3. The posterior pdf, where the targets are at -20 and 45 degree (black line).

Fig. 4. Average of MSE over the step (from 200 samples).

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