Robust Control Design for Discrete Chaotic Systems with Unmatched Uncertainties

Jason Sheng-Hong Tsai^a, Jiunn-Shiou Fang^a, Jun-Juh Yan^{b,*}, Ming-Cheng Dai^a, Shu-Mei Guo^c, Leang-San Shieh^d

^a Department of Electrical Engineering, National Cheng-Kung University, Tainan 701, Taiwan, R.O.C.

^b Department of Computer and Communication, Shu-Te University, Kaohsiung 824, Taiwan, R.O.C.

^c Department of Computer Science and Information Engineering, National Cheng-Kung University, Tainan 701, Taiwan, R.O.C.

^d Department of Electrical and Computer Engineering, University of Houston, Houston, TX 77204-4005, U.S.A.

*Corresponding Author: jjyan@stu.edu.tw

Abstract

This paper is concerned with the robust control for a general class of discrete chaotic systems with unmatched uncertainties. It is implemented by using discrete sliding mode control (DSMC). The proposed DSMC ensures the existence of the sliding mode. The chaotic behavior can be inhibited in an estimated bound in the state space even with unmatched uncertainties, which is not well addressed in the literature. The numerical simulation results demonstrate the feasibility and effectiveness of the proposed scheme.

Keywords: Discrete sliding mode control, General chaotic systems, Unmatched uncertainties.

1. Introduction

The chaotic system is with a bounded unstable dynamic behavior that exhibits sensitivity dependent on initial conditions and contains random responses in the strange attractor. However, the chaos behavior in some engineering system is highly unexpected because it will influence performance of system. In recent years, growing interests from physics, chemistry, biology, and engineering have stimulated the studies of chaos control techniques [1-6]. Several control methods leading to suppression of chaos have been presented. One of the famous masterpieces is a kind of feedback control OGY method [4], which uses some weak feedback control to make the chaotic trajectory approach and settle down finally to a desired stabilized periodic orbit, formerly unstably embedded in the chaotic manifold. The other one is a kind of non-feedback control, which usually uses given external or parametric excitations

to control the behavior of a system [6-7]. Meanwhile due to the progress of computer and digital signal processing technology, the discrete-time model of chaotic systems can be easily obtained [8] and research in discrete-time control has become intensified in recent years [8-10].

As for robust control, in these three decades, sliding mode control has become the useful and distinctive robust control strategy for controlled systems [11]. There is a main advantage of sliding mode control which is the robustness to matched uncertainties. However, sometimes in conventional sliding mode control, it is difficult to discuss the stability of the systems with unmatched uncertainties.

In this paper, we study the design of chaos suppression for the general discrete chaotic systems with unmatched uncertainties. We propose a novel DSMC scheme to solve the chaos suppression problem. This proposed DSMC scheme can be effectively applied for all general classes of chaotic systems. A special type of sliding surface is proposed to avoid the reducing order and it becomes easy to estimate the stability of the closed-loop system in sliding motion. The proposed DSMC ensures the occurrence of the discrete sliding mode and the chaotic behavior can be suppressed in a predictable bound in the state space.

2. System description and controller design

To describe the chaotic systems in the real physical world, the considered controlled discrete chaotic system with general form is assumed to be subjected to external disturbance and it can be written as follows:

$$x(k+1)T = Ax(kT) + Bg(x(kT)) + Bu(kT) + d(kT) \quad (1)$$

where T is the sampling time and $d(kT) \in R^n$ is the external disturbance bounded by γ , i.e. $\|d(k)\| \leq \gamma$ and $u(kT) \in R^m$ is the control input. For simplicity, we omit T (the sampling time) in the following discussion.

In this paper, the goal of design is to propose a DSMC such that the chaotic behavior of systems can be robustly suppressed. As a sequence, the DSMC design is composed of two phases. First, it needs to select an appropriate sliding surface such that the sliding motion on the manifold can inhibit the controlled state to a predictable bound in the state space. Second, it needs to determine a control law to guarantee the existence of the sliding mode and maintain the system dynamics on the sliding manifold [11]. To complete the design steps above, both the sliding surface and DSMC design are discussed as follows:

Phase 1: The sliding surface design:

To ensure the sliding motion on the manifold can inhibit the controlled state. The sliding surface $s(k)$ is given as:

$$s(k) = Cx(k) - C[A + BK]x(k-1) \quad (2)$$

where $s(k) \in R^m$, $C \in R^{m \times n}$ is selected such that $(CB)^{-1}$ exists and $K \in R^{m \times n}$ is selected such that the eigenvalues of $A + BK$ are all in the unit circle. Assume the system is in the sliding manifold, i.e. $s(k) = 0$ and $s(k+1) = 0$, the following relation can be obtained:

$$\begin{aligned} s(k+1) &= Cx(k+1) - C[A + BK]x(k) \\ &= CBg(x(k)) + CBu(k) + Cd(k) - CBKx(k) \quad (3) \\ &= 0 \end{aligned}$$

Therefore we can obtain the equivalent controller from (3),

$$u_{eq} = -g(x(k)) - (CB)^{-1}Cd(k) + Kx(k) \quad (4)$$

Substituting (4) into (1), we can obtain the equation in the sliding manifold as following:

$$x(k+1) = (A + BK)x(k) + (I - B(CB)^{-1}C)d(k) \quad (5)$$

For simplicity, we rewrite (5) as

$$x(k+1) = \tilde{A}x(k) + \tilde{d}(k) \quad (6)$$

where $\tilde{A} = A + BK$, $\tilde{d}(k) = (I - B(CB)^{-1}C)d(k)$.

Let $x(k) = \hat{T}\hat{x}(k)$, (6) can be rewritten as:

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{d}(k) \quad (7)$$

where $\hat{T}$ is the eigenvectors of $A + BK$, $\hat{A} = \hat{T}^{-1}\tilde{A}\hat{T}$ is the diagonal matrix form of $\tilde{A}$ and $\hat{d} = \hat{T}^{-1}\tilde{d}$.

According to (7), we can obtain the following inequality (8) when $k \rightarrow \infty$.

$$\begin{aligned} \lim_{k \rightarrow \infty} \|\hat{x}(k+1)\| &\leq \lim_{k \rightarrow \infty} \left\| \hat{A}^{k+1}\hat{x}(0) + \sum_{j=0}^k \hat{A}^j \hat{d}(k-j) \right\| \\ &\leq \lim_{k \rightarrow \infty} \|\hat{A}^{k+1}\hat{x}(0)\| + \lim_{k \rightarrow \infty} \sum_{j=0}^k \|\hat{A}^j \hat{d}(k-j)\| \end{aligned} \quad (8)$$

Here, $\lim_{k \rightarrow \infty} \|\hat{A}^{k+1}\| = 0$ caused by choosing K such that the eigenvalues of $\tilde{A}$ are all in the unit circle. Then, (8) can be rewritten as:

$$\begin{aligned} \lim_{k \rightarrow \infty} \|\hat{x}(k+1)\| &\leq \lim_{k \rightarrow \infty} \sum_{j=0}^k \|\hat{A}^j \hat{d}(k-j)\| \\ &\leq \lim_{k \rightarrow \infty} \sum_{j=0}^k \|\hat{A}^j\| \cdot \max_j \|\hat{d}(k-j)\| \\ &= (1 + \|\hat{A}\| + \|\hat{A}^2\| + \dots + \|\hat{A}^k\|) \|\hat{T}^{-1}\| \cdot \|I - B(CB)^{-1}C\| \gamma \\ &= \frac{\|\hat{T}^{-1}\| \cdot \|I - B(CB)^{-1}C\| \gamma}{1 - \|\hat{A}\|} \end{aligned} \quad (9)$$

In (9), the relation of $\|\hat{A}^k\| = \|\hat{A}\|^k$ for a diagonal matrix has been introduced. Finally, we can easily conclude the following estimated bound,

$$\begin{aligned} \lim_{k \rightarrow \infty} \|x(k+1)\| &= \lim_{k \rightarrow \infty} \|\hat{T}\hat{x}(k+1)\| \\ &\leq \lim_{k \rightarrow \infty} \|\hat{T}\| \cdot \|\hat{x}(k+1)\| \\ &\leq \frac{\|\hat{T}\| \cdot \|\hat{T}^{-1}\| \cdot \|I - B(CB)^{-1}C\| \gamma}{1 - \|\hat{A}\|} \end{aligned} \quad (10)$$

Remark 1: If the external disturbance is matched, then the disturbance $d(k)$ can be rewritten as:

$$d(k) = Bd_1(k) \quad (11)$$

According to (4), we can obtain the equivalent controller (12) when the system dynamics on the sliding manifold.

$$u_{eq} = -g(x(k)) - d_1(k) + Kx(k) \quad (12)$$

Substituting (12) into (1), the equivalent dynamic system is obtained as

$$x(k+1) = (A + BK)x(k) \quad (13)$$

In this case with matched disturbances, we can select appropriate K such that the eigenvalues of $A + BK$ are all in the unit circle. If this condition is satisfied, the system is asymptotically stable. Therefore the effect of matched disturbances can be fully eliminated in the sliding manifold.

Phase 2: Discrete Sliding Mode Controller Design

Although we have guaranteed the stability of state dynamics in the sliding manifold, we still need a discrete sliding mode controller to ensure the existence of the sliding motion. A reaching law is given in Lemma 1 [10].

Lemma 1: If the following hitting condition (14) of sliding motion is satisfied, then the trajectories of the controlled dynamics system converge to the sliding mode, i.e.

$$s(k) = 0,$$

$$\begin{cases} s(k+1) - s(k) \leq -qTs(k) - \varepsilon T \text{sign}(s(k)), & \text{if } s(k) > 0 \\ s(k+1) - s(k) \geq -qTs(k) - \varepsilon T \text{sign}(s(k)), & \text{if } s(k) < 0 \end{cases} \quad (14)$$

where $q > 0$, $\varepsilon > 0$, $1 - qT > 0$, $T > 0$ and T is the sample period.

Theorem 1: If the control input $u(k)$ is suitably designed as:

$$u(k) = -g(x(k)) + Kx(k) + (CB)^{-1}(-\rho\gamma \text{sign}(s) + s(k) - qTs(k) - \varepsilon T \text{sign}(s(k))) \quad (15)$$

where $\rho = \text{diag}[\rho_1 \ \rho_2 \ \dots \ \rho_m] \in R^{m \times m}$, $\rho_i = \| [C_{i1} \ C_{i2} \ \dots \ C_{in}] \|^2$,

then reaching condition (14) of the sliding mode is satisfied and the trajectories of the controlled dynamics system converge to the sliding mode, i.e. $s(k) = 0$.

Proof: From (14), we have

$$s(k+1) - s(k) = CBg(x(k)) + CBu(k) + Cd(k) - CBKx(k) - s(k) \quad (16)$$

Substituting (15) into (16) gives

$$s(k+1) - s(k) = -\rho\gamma \text{sign}(s(k)) - Cd(k) - qTs(k) - \varepsilon T \text{sign}(s(k))$$

Obviously

$$\begin{cases} -\rho\gamma \text{sign}(s(k)) - Cd(k) < 0, & \text{if } s(k) > 0 \\ -\rho\gamma \text{sign}(s(k)) - Cd(k) > 0, & \text{if } s(k) < 0 \end{cases} \quad (17)$$

Therefore, the hitting condition (14) is always satisfied and $s(k)$ can converge to the sliding surface $s(k) = 0$. Hence the proof is achieved completely.

3. Experimental Results

In what follows, the proposed DSMC is used to copy with the chaos suppression problem of the following Lorenz chaotic system. The continuous type of system dynamics is given as [12]

Continuous Lorenz chaotic system:

$$\dot{x}(t) = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} g(x, t) \quad (18)$$

$$\text{where } g(x, t) = \begin{bmatrix} -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix}.$$

The discrete time representation of system (18) with sample and hold process is gives by $T = 0.01$ sec. Then according to the approach in [8], the discrete-time model of system (18) is given as follows:

Discrete Lorenz chaotic system:

$$x(k+1)T = \begin{bmatrix} 0.9179 & 0.0951 & 0 \\ 0.2663 & 1.0035 & 0 \\ 0 & 0 & 0.9737 \end{bmatrix} x(kT) + \begin{bmatrix} 0.0005 & 0 \\ 0.01 & 0 \\ 0 & 0.0099 \end{bmatrix} g(x(kT)) \quad (19)$$

The chaotic figures of discretized Lorenz chaotic system (19) are shown in Fig. 1 and display strange attractors as the nominal continuous Lorenz chaotic system (18).

Fig. 1 The attractors of discrete Lorenz chaotic system.

In this example, we add unmatched external disturbances in discrete Lorenz chaotic system (19) to demonstrate the effectiveness of the proposed scheme. We will suppose that the external disturbances $d(k) = \gamma [\sin kT \ 0 \ \cos kT]^T$, bounded by $\|d(k)\| \leq \gamma$, $\gamma = 0.01$. According to (10) we can obtain the bound $\lim_{k \rightarrow \infty} \|x(k+1)\| \leq 0.167$. Then the chaotic behavior can be inhibited in an estimated bound in the state space. In the following, we give the design steps.

Step1: $K = \begin{bmatrix} -34.5661 & -35.0146 & 0 \\ 0 & 0 & -24.7934 \end{bmatrix}$ is

chosen such that the eigenvalues of matrix $A + BK$ are $\lambda = (0.873 \ 0.6818 \ 0.729)$ and $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to result

in a stable sliding motion. Therefore, the sliding surface is obtained as (2).

Step2: According to (15), the sliding mode control law is obtained as follows:

$$u(k) = -g(x(k)) + Kx(k) + (CB)^{-1}(\rho\gamma \begin{bmatrix} \frac{s_1(k)}{|s_1(k)| + \delta} \\ \frac{s_2(k)}{|s_2(k)| + \delta} \end{bmatrix} + s(k) - qTs(k) - \varepsilon T \begin{bmatrix} \frac{s_1(k)}{|s_1(k)| + \delta} \\ \frac{s_2(k)}{|s_2(k)| + \delta} \end{bmatrix}) \quad (20)$$

where $\varepsilon=1$, $q=3$, $\delta=0.01$. In the above controller, to avoid the chattering, we use the saturation function to replace the sign function [11]. When we choose δ enough small, $\text{sign}(s(k))$ can be approximated by saturation function as shown in (20)

The simulation results are shown in Figs. 2-3. From Fig. 2, it is obvious that the controller (20) can suppress the state responses of controlled system in the estimated bound as predicted. Fig. 3 displays that the sliding mode surface converges to the designed switching surface under the proposed control (20).

Fig. 2 The effectiveness of chaos suppression.

Fig. 3 Time responses of $s(kT)$.

4. Conclusions

By applying discrete sliding mode control, the problem of chaos suppression for general classes of discrete chaotic systems has been studied. A sliding surface is first proposed, and then based on it, a discrete sliding mode controller is derived to guarantee the inhibition of chaotic systems. Furthermore, the illustrative example has demonstrated the validity of the proposed theoretical results.

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