A Statistical Sensitivity Analysis Method Using Weighted Empirical Distribution Function

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Abstract

In order to estimate the q -quantile for the random variable following an unknown probability distribution, the empirical distribution function is composed from samples obtained by Monte Carlo simulation. However, the multiple sampling of Monte Carlo simulation is time-consuming in many real-world problems. Therefore, this paper proposes a new method of the computational statistics called the weighted empirical distribution function that can be used to estimate the above q -quantile from relatively few samples. The proposed method is demonstrated through a test problem and a real-world problem, namely, the sensitivity analysis of an impedance matching circuit for SAW filter.

Keywords: Computational statistics, empirical distribution, prediction interval, sensitivity analysis, SAW filter.

1. Introduction

In order to make an estimate, forecast or decision in real-world problems, a wide range of uncertainties have to be taken into account. Sensitivity analysis is the study of how the uncertainty in the output of a model can be apportioned to different sources of uncertainties in the model input⁽¹⁾. In the engineering field, various sensitivity analysis methods have been proposed⁽²⁻⁴⁾. In the traditional sensitivity analysis methods, the sensitivity is defined based on the derivative $\partial y / \partial x$ where y and x denote the output (or response) and the input (or parameter) of a model. Actually, the derivative is often approximated by using the differentiation. The traditional sensitivity analysis methods have some drawbacks. For example, the model is needed to be linearized. Besides, the sensitivity is represented by the

relative ratio but not measured by the absolute amount.

The statistical sensitivity analysis is a novel method that utilizes the computer algorithms of the computational statistics⁽⁵⁾ for analyzing a model statistically. Besides, the Monte Carlo simulation is fundamental for the statistical sensitivity analysis. In the statistical sensitivity analysis method, the sensitivity is measured by the probability.

In this paper, the relationship between the probabilistic input and output variables of a model is considered. It is supposed that the input of the model is a set of random variables following an established probability distribution, while the model output is a random variable following an unknown probability distribution. In order to estimate the q -quantile of the latter random variable, or the output, the Empirical Cumulative Distribution Function (ECDF)⁽⁵⁾ is composed from a set of samples obtained by Monte Carlo simulation. The ECDF is a proper method to approximate an unknown Cumulative Distribution Function (CDF) from samples. The accuracy of ECDF depends on the number of samples. However, the multiple sampling based on Monte Carlo simulation is very time-consuming in many real-world problems. Therefore, in order to approximate an unknown CDF from relatively few samples, this paper proposes a new method named the Weighted Empirical Cumulative Distribution Function (W_ECDF).

The rest of this paper is organized as follows. In Section 2, the conventional ECDF⁽⁵⁾ and the proposed W_ECDF are described. In order to demonstrate the performance of the W_ECDF, it is compared with the ECDF through a test problem in Section 3. In Section 4, the proposed method is applied to a real-world problem, namely, the sensitivity analysis of an impedance matching circuit for Surface Acoustic Wave (SAW) filter. Section 5 presents conclusions and some insights to future work.

Fig. 1. Example of ECDF and smoothed ECDF

Fig. 2. Example of W-ECDF and smoothed W-ECDF

2. Statistical Sensitivity Analysis

2.1 Problem Formulation

As mentioned above, the relationship between the probabilistic input and output variables of a model is considered. Let $\vec{X} \in R^D$ be a vector of random variables following an established probability distribution. The vector of random variables is an input of the considering model that can be represented by a function $h: R^D \rightarrow R$ as

$$Y = h(\vec{X}) = h(X_1, X_2, \dots, X_D). \quad (1)$$

The output of the model denoted by Y becomes a random variable, too. The CDF of Y is defined as

$$F(y) = \Pr(Y \leq y). \quad (2)$$

From the inverse function of the CDF in (2), we can obtain the accurate q-quantile ($0 \leq q \leq 1$) of Y as

$$y_q = F^{-1}(q). \quad (3)$$

Besides, for a significance level α ($0 \leq \alpha \leq 1$), we can evaluate the prediction interval of the random variable Y by using $\alpha/2$ and $(1-\alpha/2)$ quantiles as

$$\Pr(F^{-1}(\alpha/2) \leq Y \leq F^{-1}(1-\alpha/2)) = 1-\alpha. \quad (4)$$

Both the quantile in (3) and the prediction interval in (4) provide useful information about the model in (1). For example, since $(1-\alpha)\%$ of the future observations of the random variable Y will fall in the prediction interval, the worst case of the output can be evaluated statistically by using the upper bound of the prediction interval⁽⁶⁾.

Unfortunately, the strict CDF shown in (2) can't be derived mathematically for many real-world problems. That is because the function $h: R^D \rightarrow R$ in (1) is usually too complex. Therefore, we need to employ a method of the computational statistics for composing the CDF of Y from a set of samples obtained by Monte Carlo simulation.

2.2 Empirical Distribution Function

The empirical distribution function, or ECDF, is the distributed function associated with the empirical measure of samples. ECDF is a step function that jumps up by $1/N$ at each of the N data points. From (1), each sample Y^n of the output Y is brought by a sample $\vec{X}^n$ of the input $\vec{X}$ such as $Y^n = h(\vec{X}^n)$ ($n=1, \dots, N$).

Thereby, the ECDF of Y is defined as follows:

$$\begin{cases} F_u(y) = \frac{1}{N} \sum_{n=1}^N u(Y^n \leq y), \\ u(Y^n \leq y) = \begin{cases} 1; & \text{if } Y^n \leq y, \\ 0; & \text{otherwise.} \end{cases} \end{cases} \quad (5)$$

The step function in (5), or ECDF, is smoothed. Let $\tilde{F}_u(y)$ be a smoothed ECDF in (5). Thereby, we can estimate the q-quantile ($0 \leq q \leq 1$) of Y as

Fig. 3. Change of 0.95-quantile for sample size

$$y_q \approx \tilde{F}_u^{-1}(q). \quad (6)$$

Figure 1 illustrates an example of the ECDF and its smoothed function composed from a set of samples of a normally distributed random variable: $Y \sim \mathbf{N}(0, 1)$.

The number of samples taken from the tail part of the probability distribution of $\vec{X}$ is small. Therefore, in order to approximate CDF in (2) by ECDF exactly, a large number of samples $\vec{X}^n$ ($n = 1, \dots, N$) are needed.

2.3 Weighted Empirical Distribution Function

Let $Y^n = h(\vec{X}^n)$ ($n = 1, \dots, N$) be a set of samples of the output Y in (1). Each of the samples $Y^n = h(\vec{X}^n)$ has its own weight that is given by the Probability Density Function (PDF) value $f(\vec{X}^n)$ of a corresponding sample of the input. Thereby, the weighted empirical distribution function, or W_ECDF, is defined as

$$\begin{cases} F_w(y) = \frac{1}{W} \sum_{n=1}^N w(Y^n \leq y), \\ W = \sum_{n=1}^N f(\vec{X}^n), \\ w(Y^n \leq y) = \begin{cases} f(\vec{X}^n); & \text{if } Y^n \leq y, \\ 0; & \text{otherwise.} \end{cases} \end{cases} \quad (7)$$

Contrary to the ECDF defined by (5), the samples $\vec{X}^n$ ($n = 1, \dots, N$) are not required to follow the probability distribution of $\vec{X}$ for the W_ECDF in (7).

Consequently, we can specify arbitrary data points for the samples $\vec{X}^n$ ($n = 1, \dots, N$) of the input.

The step function in (7), or W_ECDF, is smoothed. Let $\tilde{F}_w(y)$ be a smoothed W_ECDF in (7). Thereby, we can estimate the q -quantile ($0 \leq q \leq 1$) of Y as

$$y_q \approx \tilde{F}_w^{-1}(q). \quad (8)$$

Figure 2 illustrates an example of the W_ECDF and its smoothed function that approximate the same CDF with Fig. 1. Figure 2 uses the same number of samples $N = 10$ with Fig. 1. But, those samples Y^n ($n = 1, \dots, N$) were given by one-dimensional Halton sequence. Besides, each sample was weighted by the PDF of Y ($Y = X$).

3. Test Problem

The performance of the proposed W_ECDF in (7) is compared with the conventional ECDF in (5) through a test problem. The model of the test problem is given as

$$Y = h(X_1, X_2) = X_1^2 + X_2^2. \quad (9)$$

The inputs of the model X_1 and X_2 are mutually independent random numbers following the standard normal distribution. Therefore, the output Y becomes the random variable following χ^2 -distribution with two degrees of freedom. From (3), the 0.95-quantile of Y is

$$y_{0.95} = F^{-1}(0.95) = 5.991. \quad (10)$$

For composing ECDF, the input points (X_1^n, X_2^n) ($n = 1, \dots, N$) were sampled from the random numbers: $X_1, X_2 \sim \mathbf{N}(0, 1)$. On the other hand, for composing W_ECDF, the input points (X_1^n, X_2^n) were given by two-dimensional Halton sequence and weighted by PDF.

Figure 3 shows the 0.95-quantiles of Y estimated by using the conventional ECDF and the proposed W_ECDF respectively, which depend on the sample size. From Fig. 3, we can confirm that W_ECDF estimates the 0.95-quantile in (10) accurately with fewer samples than ECDF.

Fig. 4. Impedance matching circuit for SAW filter

Fig. 5. Frequency response of SAW filter

Fig. 6. Smith chart of SAW filter

4. Sensitivity Analysis of SAW Filter

4.1 Impedance Matching Circuit for SAW Filter

Surface Acoustic Wave (SAW) devices⁽⁷⁾ including SAW filter and SAW duplexer are widely used in the modern RF circuits of various mobile communication systems such as cellular phones. SAW devices have

achieved world-wide success as mobile communication technology over the last several decades. Besides, they are studied actively to this day⁽⁸⁻¹⁰⁾. Author has been studying the computer-aided design methods of SAW filters for many years⁽¹¹⁻¹³⁾. The proposed method is used to analyze an impedance matching circuit for SAW filter in Fig. 4.

4.2 Network Model of SAW Filter

The equivalent circuit model of the SAW filter shown in Fig. 4 is described by transmission line parameters:

$(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ ⁽¹⁴⁾. From those parameters, the transmission

line parameters (A, B, C, D) of the SAW filter with the impedance matching circuit in Fig. 4 can be derived as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{j\omega C_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega L_1} & 0 \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \frac{1}{j\omega C_2} & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \quad (11)$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

where V_1 and I_1 are the voltage and current at the input port of the impedance matching circuit, while V_2 and I_2 are the voltage and current at the output port as shown in Fig. 4.

The transmission line parameters derived in (11) can be transformed into scattering parameters as

$$\begin{cases} S_{11} = \frac{AZ_2 + B - CZ_1Z_2 - DZ_1}{AZ_2 + B + CZ_1Z_2 + DZ_1}, \\ S_{21} = \frac{2\sqrt{Z_1Z_2}}{AZ_2 + B + CZ_1Z_2 + DZ_1}, \\ S_{12} = \frac{2(AD - BC)\sqrt{Z_1Z_2}}{AZ_2 + B + CZ_1Z_2 + DZ_1}, \\ S_{22} = \frac{-AZ_2 + B - CZ_1Z_2 + DZ_1}{AZ_2 + B + CZ_1Z_2 + DZ_1}. \end{cases} \quad (12)$$

By using the scattering parameters in (12), the network model of the SAW filter including the impedance matching circuit in Fig. 4 can be described as

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (13)$$

where a_1 and b_1 denote the input and output signals at the input port of the circuit, while a_2 and b_2 denote the input and output signals at the output port of the circuit⁽⁷⁾.

The scattering parameters S_{21} in (13) provides the transition characteristic from the input port to the output port. Then the insertion loss of SAW filter is defined as

Fig. 7. Prediction interval with ECDF ($k=0.05$, $N=100$)

Fig. 9. Prediction interval with W_ECDF ($k=0.05$, $N=100$)

Fig. 8. Prediction interval with ECDF ($k=0.1$, $N=100$)

Fig. 10. Prediction interval with W_ECDF ($k=0.1$, $N=100$)

$$Loss = 20 \log_{10}(|S_{21}(\omega, L_1, C_1, L_2)|). \quad (14)$$

The value S_{21} depends on the frequency ω and the three electric elements of the impedance matching circuit: the inductances L_1 and L_2 , and the capacitance C_1 . Figure 5 shows the frequency response of the SAW filter, or the insertion loss in (14), with nominal values of L_1 , C_1 , and L_2 shown in Fig. 4. Incidentally, Fig. 6 shows the Smith chart of the SAW filter in which the blue line denotes the input-port and the red line denotes the output-port.

Actually, the values of the three electric elements, namely L_1 , C_1 , and L_2 , fluctuate due to environmental condition and processing error. Therefore, we try to analyze the effect of the perturbations in the electric elements on the insertion loss in (14). Strictly speaking, the formula in (14) is extended into an instance of the model in (1). We suppose that the values of the three electric elements are mutually independent normal random variables as follows:

$$\begin{cases} L_1 & \sim \mathbf{N}(L_1^*, (k L_1^*)^2), \\ C_1 & \sim \mathbf{N}(C_1^*, (k C_1^*)^2), \\ L_2 & \sim \mathbf{N}(L_2^*, (k L_2^*)^2), \end{cases} \quad (15)$$

where $L_1^*=56\text{nH}$, $C_1^*=18\text{pF}$, and $L_2^*=82\text{nH}$.

4.3 Result of Statistical Sensitivity Analysis

From the three random variables given by (15), the prediction interval defined in (4) was calculated with a significance level $\alpha = 0.1$ for the insertion loss in (14).

Figures 7 and 8 show the prediction intervals of the insertion loss obtained by using ECDF with $N=100$ for different variance factors, namely $k=0.05$ and $k=0.1$, at each frequency point ω in (14). Similarly, Figs. 9 and 10 show the prediction intervals obtained by using W_ECDF. For W_ECDF, the input points (L_1^n, C_1^n, L_2^n) ($n=1, \dots, N$) were given by three-dimensional Halton sequence.

From Figs. 7 and 9, W_ECDF provides wider prediction intervals than ECDF for the same variance factor $k=0.05$. Similarly, from Figs. 8 and 10, W_ECDF provides wider prediction intervals than ECDF for the same variance factor $k=0.1$. Especially, the lower bounds of the prediction intervals with W_ECDF are falling below the lower bounds with ECDF. That is because ECDF can't get enough number of samples (L_1^n, C_1^n, L_2^n) from the tail part of the three-dimensional normal distribution shown in (15).

Comparing Fig. 9 with Fig. 10, we can confirm that the uncertainty of the insertion loss increases in proportion to the amount of the perturbations in the electric elements.

5. Conclusions

In order to estimate the q-quantile for the random variable following an unknown probability distribution, a new method named W_ECDF was proposed. From the results of experiments conducted on a test problem and a real-world problem, W_ECDF outperformed ECDF.

In our future work, a multi-dimensional W_ECDF will be developed. If we use a two-dimensional W_ECDF, we may draw the prediction interval even on the Smith chart in Fig. 6. Furthermore, we will use the q-quantile obtained by the proposed W_ECDF to measure the robustness of the solutions for various robust optimization problems⁽⁶⁾⁽¹⁵⁾.

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