Robust Control Design for Discrete Chaotic Systems with Unmatched Uncertainties

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where $T$ is the sampling time and $d(kT) \in \mathbb{R}^n$ is the external disturbance bounded by $\gamma$, i.e. $\|d(k)\| \leq \gamma$ and $u(kT) \in \mathbb{R}^m$ is the control input. For simplicity, we omit $T$ (the sampling time) in the following discussion.

In this paper, the goal of design is to propose a DSMC such that the chaotic behavior of systems can be robustly suppressed. As a sequence, the DSMC design is composed of two phases. First, it needs to select an appropriate sliding surface such that the sliding motion on the manifold can inhibit the controlled state to a predictable bound in the state space. Second, it needs to determine a control law to guarantee the existence of the sliding mode and maintain the system dynamics on the sliding manifold \[11\]. To complete the design steps above, both the sliding surface and DSMC design are discussed as follows:

**Phase 1: The sliding surface design:**

To ensure the sliding surface of the manifold can inhibit the controlled state. The sliding surface $s(k)$ is given as:

$$s(k) = Cx(k) - C[A + BK]x(k-1)$$  
(2)

where $s(k) \in \mathbb{R}^n , C \in \mathbb{R}^{m \times n}$ is selected such that $(CB)^{-1}$ exists and $K \in \mathbb{R}^{m \times n}$ is selected such that the eigenvalues of $A + BK$ are all in the unit circle. Assume the system is in the sliding manifold, i.e. $s(k) = 0$ and $s(k+1) = 0$, the following relation can be obtained:

$$s(k+1) = Cx(k+1) - C[A + BK]x(k)$$
$$= CBg(x(k)) + CBu(k) + Cd(k) - CBx(k)$$

$$= 0$$

Therefore we can obtain the equivalent controller from (3),

$$u_d = -g(x(k)) - (CB)^{-1}Cd(k) + Kx(k)$$  
(4)

Substituting (4) into (1), we can obtain the equation in the sliding manifold as following:

$$x(k+1) = (A + BK)x(k) + (I - B(CB)^{-1}C)d(k)$$  
(5)

For simplicity, we rewrite (5) as

$$x(k+1) = \hat{A}x(k) + \hat{d}(k)$$  
(6)

where $\hat{A} = A + BK, \hat{d}(k) = (I - B(CB)^{-1}C)d(k)$. Let

$$x(k) = \hat{T}\hat{x}(k),$$

(6) can be rewritten as:

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{d}(k)$$  
(7)

where $\hat{T}$ is the eigenvectors of $A + BK, \hat{A} = \hat{T}^{-1}\hat{A}\hat{T}$ is the diagonal matrix form of $\hat{A}$ and $\hat{d} = \hat{T}^{-1}\hat{d}$.

According to (7), we can obtain the following inequality (8) when $k \rightarrow \infty$.

$$\lim_{k \rightarrow \infty} \|\hat{x}(k+1)\| \leq \lim_{k \rightarrow \infty} \|\hat{A}\hat{x}(k)+\hat{d}(k)\|$$
$$\leq \lim_{k \rightarrow \infty} \|\hat{A}\hat{x}(0)\| + \lim_{k \rightarrow \infty} \sum_{j=0}^{\infty} \|\hat{d}(k-j)\|$$

(8)

Here, $\lim_{k \rightarrow \infty} \|\hat{A}\hat{x}(k)\| = 0$ caused by choosing $K$ such that the eigenvalues of $\hat{A}$ are all in the unit circle. Then, (8) can be rewritten as:

$$\lim_{k \rightarrow \infty} \|\hat{x}(k+1)\| \leq \lim_{k \rightarrow \infty} \sum_{j=0}^{\infty} \|\hat{d}(k-j)\|$$
$$\leq \lim_{k \rightarrow \infty} \sum_{j=0}^{\infty} \|\hat{A}\hat{x}(j)\| + \lim_{k \rightarrow \infty} \sum_{j=0}^{\infty} \|\hat{d}(k-j)\|$$

(9)

In (9), the relation of $\|\hat{A}\| \leq \|\hat{d}\|$ for a diagonal matrix has been introduced. Finally, we can easily conclude the following estimated bound,

$$\lim_{k \rightarrow \infty} \|\hat{x}(k+1)\| = \lim_{k \rightarrow \infty} \|\hat{T}\hat{x}(k+1)\|$$
$$\leq \|\hat{T}\| \|\hat{x}(k+1)\|$$
$$\leq \|\hat{T}\| \|\hat{x}(k)\|$$

(10)

**Remark 1:** If the external disturbance is matched, then the disturbance $d(k)$ can be rewritten as:

$$d(k) = Bd_d(k)$$  
(11)

According to (4), we can obtain the equivalent controller (12) when the system dynamics on the sliding manifold.

$$u_d = -g(x(k))-d_d(k) + Kx(k)$$

(12)

Substituting (12) into (1), the equivalent dynamic system is obtained as

$$x(k+1) = (A + BK)x(k)$$

(13)

In this case with matched disturbances, we can select appropriate $K$ such that the eigenvalues of $A + BK$ are all in the unit circle. If this condition is satisfied, the system is asymptotically stable. Therefore the effect of matched disturbances can be fully eliminated in the sliding manifold.

**Phase 2: Discrete Sliding Mode Controller Design**

Although we have guaranteed the stability of state dynamics in the sliding manifold, we still need a discrete sliding mode controller to ensure the existence of the sliding motion. A reaching law is given in Lemma 1 \[10\].
Lemma 1: If the following hitting condition (14) of sliding motion is satisfied, then the trajectories of the controlled dynamics system converge to the sliding mode, i.e. $s(k) = 0$.

$$
\begin{cases}
(s(k+1) - s(k)) \leq -qTs(k) - \varepsilon T \text{sign}(s(k)), & \text{if } s(k) > 0 \\
(s(k+1) - s(k)) \geq -qTs(k) - \varepsilon T \text{sign}(s(k)), & \text{if } s(k) < 0
\end{cases}
$$

(14)

where $q > 0$, $\varepsilon > 0$, $1 - qT > 0$, $T > 0$ and $T$ is the sampling period.

Theorem 1: If the control input $u(k)$ is suitably designed as:

$$
u(k) = -g(x(k)) + Ks(k) + (CB)^T(-\rho \varepsilon \text{sign}(s(k)) - qTs(k) - \varepsilon T \text{sign}(s(k)))
$$

(15)

where $\rho = \text{diag} \{ \rho_1, \rho_2, ..., \rho_n \} \in \mathbb{R}^{n \times n}$, $\rho = \begin{bmatrix} C_{11} & C_{12} & ... & C_{1n} \end{bmatrix}$, then reaching condition (14) of the sliding mode is satisfied and the trajectories of the controlled dynamics system converge to the sliding mode, i.e. $s(k) = 0$.

Proof: From (14), we have

$$
s(k+1) - s(k) = CBg(x(k)) + CBa(x(k)) + Cd(x(k)) - CBKx(k) - s(k)
$$

(16)

Substituting (15) into (16) gives

$$s(k+1) - s(k) = -\rho \varepsilon \text{sign}(s(k)) - Cd(x(k)) - qTs(k) - \varepsilon T \text{sign}(s(k))$$

Obviously

$$
\begin{cases}
-\rho \varepsilon \text{sign}(s(k)) - Cd(x(k)) < 0, & \text{if } s(k) > 0 \\
-\rho \varepsilon \text{sign}(s(k)) - Cd(x(k)) > 0, & \text{if } s(k) < 0
\end{cases}
$$

(17)

Therefore, the hitting condition (14) is always satisfied and $s(k)$ can converge to the sliding surface $s(k) = 0$. Hence the proof is achieved completely.

3. Experimental Results

In what follows, the proposed DSMC is used to copy with the chaos suppression problem of the following Lorenz chaotic system. The continuous type of system dynamics is given as [12]

**Continuous Lorenz chaotic system:**

$$
\dot{x}(t) =
\begin{bmatrix}
-10 & 10 & 0 \\
28 & -1 & 0 \\
0 & 0 & -\frac{8}{3}
\end{bmatrix}
x(t) +
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}
g(x,t)
$$

(18)

where $g(x,t) = [-x_1(t)x_2(t) x_1(t)x_2(t)]$.

The discrete time representation of system (18) with sample and hold process is given by $T = 0.01$ sec. Then according to the approach in [8], the discrete-time model of system (18) is given as follows:

**Discrete Lorenz chaotic system:**

$$
x(k+1)T =
\begin{bmatrix}
0.9179 & 0.0951 & 0 \\
0.2663 & 1.0035 & 0 \\
0 & 0 & 0.0099
\end{bmatrix}
x(kT)
+ \begin{bmatrix}
0.0005 \\
0.01 \\
0
\end{bmatrix}
g(x(kT))
$$

(19)

The chaotic figures of discretized Lorenz chaotic system (19) are shown in Fig. 1 and display strange attractors as the nominal continuous Lorenz chaotic system (18).

![Fig. 1 The attractors of discrete Lorenz chaotic system.](image)

In this example, we add unmatched external disturbances in discrete Lorenz chaotic system (19) to demonstrate the effectiveness of the proposed scheme. We will suppose that the external disturbances $d(k) = \gamma \begin{bmatrix} \sin kT & 0 & \cos kT \end{bmatrix}$, bounded by $\|d(k)\| \leq \gamma$. According to (10) we can obtain the bound $\lim_{k \to \infty} \|x(k+1)\| \leq 0.167$. Then the chaotic behavior can be inhibited in an estimated bound in the state space. In the following, we give the design steps.

**Step1:** $K = \begin{bmatrix} -34.5661 & -35.0146 & 0 \\
0 & 0 & -24.7934\end{bmatrix}$ is chosen such that the eigenvalues of matrix $A + BK$ are $\lambda = (0.873, 0.6818, 0.729)$ and $C = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1\end{bmatrix}$ to result in a stable sliding motion. Therefore, the sliding surface is obtained as (2).
Step2: According to (15), the sliding mode control law is obtained as follows:
\[
\begin{align*}
\dot{s}(k) &= -g(s(k)) + K \dot{s}(k) + \varepsilon (s(k) - q T s(k)) - \delta T s(k) \\
+ (CB)^T \rho T & \begin{bmatrix}
\tilde{s}_1(k) & \tilde{s}_2(k) & \tilde{s}_3(k) & \tilde{s}_4(k)
\end{bmatrix}
\end{align*}
\]
where \( \varepsilon = 1, q = 3, \delta = 0.01 \). In the above controller, to avoid the chattering, we use the saturation function to replace the sign function [11]. When we choose \( \delta \) enough small, \( \text{sign}(s(k)) \) can be approximated by saturation function as shown in (20).

The simulation results are shown in Figs. 2-3. From Fig. 2, it is obvious that the controller (20) can suppress the state responses of controlled system in the estimated bound as predicted. Fig. 3 displays that the sliding mode surface converges to the designed switching surface under the proposed control (20).

![Figure 2](image1.png)  
**Fig. 2** The effectiveness of chaos suppression.

![Figure 3](image2.png)  
**Fig. 3** Time responses of \( s(kT) \).

4. Conclusions

By applying discrete sliding mode control, the problem of chaos suppression for general classes of discrete chaotic systems has been studied. A sliding surface is first proposed, and then based on it, a discrete sliding mode controller is derived to guarantee the inhibition of chaotic systems. Furthermore, the illustrative example has demonstrated the validity of the proposed theoretical results.

Acknowledgments

The authors gratefully acknowledge the support of Ministry of Science and Technology of Taiwan through the grant MOST 105-2221-E-366-004.

References